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Acceleration of propagation in a chemotaxis-growth system with slowly decaying initial data

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Abstract

In this paper, we study the spatial propagation dynamics of a parabolic–elliptic chemotaxis system with logistic source which reduces to the well-known Fisher-KPP equation without chemotaxis. It is known that for fast decaying initial functions, this system has a finite spreading speed. For slowly decaying initial functions, we show that the accelerating propagation will occur and chemotaxis does not affect the propagation mode determined by slowly decaying initial functions if the logistic damping is strong, that is, the system has the same upper and lower bounds of the accelerating propagation as for the classical Fisher-KPP equation. The main new idea of proving our results is the construction of auxiliary equations to overcome the lack of comparison principle due to chemotaxis.

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1 | INTRODUCTION

Chemotaxis, as a strategy of migration, describes the directional movement of cells along a chemical concentration gradient. It was well known that this process can promote the rapid propagation of bacterial populations into previously unoccupied territories (cf. [1, 2, 8, 36, 51]). The first mathematical model was attributed to Patlak [45]. The propagation of migrating bands of bacterial chemotaxis was first observed in the experiment by Adler [1] and the following mathematical model was proposed by Keller and Segel [25] to recover the migrating bands of bacterial chemotaxis using a pair of reaction–diffusion–convection equations (nowadays well known as singular Keller–Segel system)

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \\ v_t = \varepsilon \Delta v - u v^m, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the bacterial density and $v(x,t)$ the chemical (oxygen) concentration at position x and at time $t > 0$, respectively. $\chi > 0$ is the chemotactic coefficient and $\varepsilon \geqslant 0$ denotes the chemical diffusivity. We remark that $m = 0$ was assumed for analysis in [25] and $m \geqslant 0$ in [24]. The Keller–Segel system (1.1) has attracted extensive studies generating a large number of beautiful mathematical results on the existence and stability of traveling wave solutions (cf. [10, 12, 24, 31, 33, 38, 41]) as well as global solvability (cf. [21, 22, 32]). In order to generate traveling bands, the Keller–Segel system (1.1) requires a singular chemotactic sensitivity for sufficiently small chemical concentration, which is, however, unrealistic since cells cannot perform chemotaxis when concentrations fall below detectable values (cf. [42]). Furthermore, this model neglected cell growth, a substantial factor in the expansion process (cf. [42]). Subsequently various models including cell growth but without singular sensitivity have been proposed and tested against numerical simulations (cf. [27, 28, 30, 54]). In this paper we are concerned with the spatial propagation dynamics of the following classical chemotaxis system with logistic growth

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + u(a - bu), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - \lambda v + \mu u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where χ, a, b, λ, and μ are positive constants, τ is a non-negative constant, $u(x,t)$ denotes the cell density, and $v(x,t)$ the chemical concentration at position x and at time $t > 0$. The system (1.2) models the movement of cells directed by the higher concentration of chemoattractant emitted from cells. The constant χ is called the chemotactic coefficient, λ is the degradation rate of chemoattractant, μ is the rate at which cells produce chemoattractant, and the constant $1/\tau$ in the case $\tau > 0$ measures the diffusion rate of chemoattractant. It has been numerically demonstrated in [44] that the model (1.2) can produce various intricate patterns including traveling waves.

For (1.2), the global solvability of solutions and traveling waves are two primary analytical research topics. In the case $\tau = 0$, it was shown in [23, 53] that on a bounded domain $\Omega \subseteq \mathbb{R}^N$ complemented with Neumann boundary conditions, (1.2) has a unique globally bounded classical solution for any non-negative initial data $u_0 \in C^{0,\alpha}(\bar{\Omega})$ with $\alpha \in (0,1)$ if either $N \leqslant 2$ or $b > \frac{N-2}{N}\chi\mu$. Under the same condition, if $\Omega = \mathbb{R}^N$, there exists a unique global classical solution of (1.2) for any non-negative initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p > N$ and $p \geqslant 2$ (cf. [48]). Similar results were obtained for the case $\tau > 0$ in [43, 56]. When $N \geqslant 3$ and $b < \frac{N-2}{N}\chi\mu$, to the best of our knowledge, whether (1.2) admits globally bounded or blow-up solutions still remains open for both bounded and unbounded domain $\Omega \subseteq \mathbb{R}^N$. For the results on traveling wave solutions, when $a = b = \lambda = \mu = 1$ and $\tau = 0$, the existence of traveling wave solutions of (1.2) was obtained in [40] for $0 < \chi < 1$ and in [46] for $0 < \chi < \frac{1}{2}$ with the existence of a minimal wave speed. If $\lambda > a$ and $b > 2\chi\mu$, (1.2) has a traveling wave solution on $\mathbb{R}$ with a minimal speed $c \geqslant 2\sqrt{a}$ connecting (0,0) and $(\frac{a}{b}, \frac{a\mu}{b\lambda})$, see [50] for $\tau = 0$ and [47, 49] for $\tau > 0$. However, the question whether the traveling wave solution is monotone and takes the values only in $(0, \frac{a}{b}) \times (0, \frac{a\mu}{b\lambda})$ remains open. When one of the conditions $\lambda > a$ and $b > 2\chi\mu$ fails, the existence of traveling wave solutions of (1.2) is also an open question.

Except the global solvability of solution and traveling waves, the spatial propagation dynamics of (1.2) is another interesting research topic and not many results are available in the literature. In this paper, we will investigate the spatial propagation dynamics of (1.2) on $\mathbb{R}$ with $\tau = 0$, namely,

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + u(a - bu), & x \in \mathbb{R},\ t > 0, \\ 0 = v_{xx} - \lambda v + \mu u, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Denote

$$C^b_{\text{unif}}(\mathbb{R}) = \{u \in C(\mathbb{R}) \mid u \text{ is uniformly continuous in } x \in \mathbb{R} \text{ and } \sup_{x\in\mathbb{R}} |u(x)| < \infty\},$$

which is equipped with the norm $\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)|$. Throughout the paper, we assume that the initial function satisfies

$$u_0 \in C^b_{\text{unif}}(\mathbb{R}),\ u_0(x) > 0 \text{ for } x \in \mathbb{R},\ \liminf_{x\to-\infty} u_0(x) > 0, \text{ and } u_0(x) \to 0 \text{ as } x \to +\infty. \tag{1.4}$$

When $b > \chi\mu$, the global existence of solution of (1.3) with $u_0 \in C^b_{\text{unif}}(\mathbb{R})$ was obtained in [48]. There are two classes of initial function u_0 that are commonly used in the literature as follows:

(a) **fast decaying initial function**, namely, there is $\beta > 0$ and $C > 0$ such that

$$u_0(x) \leqslant Ce^{-\beta x} \text{ for large } x, \tag{1.5}$$

(b) **slowly decaying initial function**, namely, there is a large constant ξ_0 such that

$$u_0 \in C^2([\xi_0, +\infty)),\ u_0' \leqslant 0 \text{ in } [\xi_0, +\infty), \text{ and } u_0''(x)/u_0(x) \to 0 \text{ as } x \to +\infty. \tag{1.6}$$

By Lemma 2.2(ii) below, when u_0 satisfies (1.4) and (1.6), u_0 decays more slowly than any exponentially decaying function as $x \to +\infty$, namely,

$$\forall\, \kappa,\ \exists\, x_\kappa \text{ s.t. } u_0(x) \geqslant e^{-\kappa x} \text{ for all } x \in [x_\kappa, +\infty).$$

In the case $\chi = 0$, u in (1.3) is independent of v, and if we ignore the second equation, the first equation of (1.3) with initial data $u_0(x)$ reduces to the following well-known Fisher-KPP equation:

$$\begin{cases} u_t = u_{xx} + u(a - bu), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

The spatial propagation dynamics of (1.7) with different initial functions has been extensively studied as one of the prevailing research topics in the past few decades. For example, for fast decaying initial functions, if $u_0(x) \leqslant c_0 e^{-\sqrt{a}x}$ with $c_0 > 0$ for large x, then (1.7) has a spreading speed $2\sqrt{a}$ (see [5, 6, 26]), and if there are two constants $c_1, c_2 \in (0, +\infty)$ such that $c_1 \leqslant u_0(x)e^{\kappa x} \leqslant$

c_2 with $\kappa \in (0, \sqrt{a})$ for large x, then (1.7) has a spreading speed $\kappa + a/\kappa$ (see [7, 17, 29, 39, 52]). Here a constant $c^* > 0$ is called a *spreading speed* if $u(x,t)$ satisfies that

$$\begin{cases} \lim\limits_{t\to\infty} \sup\limits_{x\geqslant ct} |u(x,t)| = 0 & \text{for any } c > c^*, \\ \liminf\limits_{t\to\infty} \inf\limits_{x\leqslant ct} u(x,t) > 0 & \text{for any } c < c^*. \end{cases}$$

We refer to [34, 35, 37, 55] for more results on the spreading speed of discrete-time recursion equations which can include (1.7) as a special example. On the other hand, for slowly decaying initial functions, as shown in [18] by Hemal and Roques, (1.7) has a new spatial propagation mode-*acceleration propagation*, which is quite different from the propagation mode of spreading speed resulting from fast decaying initial functions. The acceleration propagation means that the average moving speed of the level set $E_\omega(t)$ tends to infinity as $t \to +\infty$, in the sense that

$$\lim_{t\to+\infty} \frac{\inf\{E_\omega(t)\}}{t} = +\infty,$$

where $E_\omega(t)$ is the moving level set defined by $E_\omega(t) = \{x \in \mathbb{R}, u(x,t) = \omega\}$. The authors of [18] also showed that for any $\gamma_1, \gamma_2 > 0$, $\epsilon \in (0, a)$, and $\omega \in (0, a/b)$, there exists $T > 0$ such that

$$E_\omega(t) \subseteq u_0^{-1}\left\{ \left[\gamma_1 e^{-(a+\epsilon)t}, \gamma_2 e^{-(a-\epsilon)t}\right] \right\} \text{ for all } t \geqslant T, \tag{1.8}$$

where $u_0^{-1}\{A\} = \{x \in \mathbb{R}, u_0(x) \in A\}$ denotes the inverse image of u_0 from the set A. By (1.8), the explicit upper and lower bounds locating $E_\omega(t)$ were obtained in [18] for different forms of u_0, for example, if $u_0(x) = Cx^{-p}$ for large x with $p, C > 0$, then $\ln(\min E_\omega(t)) \sim \ln(\max E_\omega(t)) \sim ap^{-1}t$ as $t \to +\infty$. We refer to [3, 19, 20] for the acceleration propagation results recently developed for more general reaction–diffusion equations. We also refer to [9, 11, 13] for the acceleration propagation of fractional diffusion equations and [4, 14, 15] for non-local dispersal equations.

Compared to the Fisher-KPP equation, the results on the spatial propagation dynamics of (1.3) with $\chi > 0$ are much less. When u_0 is a fast decaying initial function, Salako and Shen studied the existence of spreading speed of (1.3) in [50]. To summarize their main results in [50], we introduce a set

$$K = \begin{cases} \emptyset, & \text{when } b < \chi\mu, \\ \left(-\infty, \dfrac{2b - \chi\mu}{3\chi\mu - 2b}\sqrt{\lambda}\right), & \text{when } \chi\mu \leqslant b < \dfrac{3}{2}\chi\mu, \\ \mathbb{R}, & \text{when } b \geqslant \dfrac{3}{2}\chi\mu. \end{cases}$$

We denote the spreading speed of (1.3) by a positive constant c^*. When $b \geqslant \chi\mu$ and u_0 is a fast decaying initial function, it was shown in [50] that $c^* = 2\sqrt{a}$ if $\sqrt{a} \in K$ and $u_0(x) = 0$ for large x, while $c^* = \kappa + a/\kappa$ if $u_0(x) \to e^{-\kappa x}$ as $x \to +\infty$ with $\kappa \in (0, \sqrt{a}) \cap K$. Compared to the results for the Fisher-KPP equation (1.7), it was found that the chemotaxis neither asymptotically speeds up nor slows down the spreading speed for fast decaying initial functions when the chemotactic sensitivity is weak (that is, $\chi \leqslant \frac{b}{\mu}$). However, the case of strong chemotaxis (that is, $\chi > \frac{b}{\mu}$) was left open.

If we solve v in terms of u by the constant of variations, we can reformulate the system (1.3) into a scalar Fisher-KPP-type equation with non-local advection as follows:

$$u_t + [(K * u)u]_x = u_{xx} + u(1-u), \tag{1.9}$$

where we have assumed $a = b = 1$ without loss of generality and

$$K(x) = -\frac{1}{2}\chi\mu\,\mathrm{sign}(x)e^{-\sqrt{\lambda}|x|},\ x \in \mathbb{R}. \tag{1.10}$$

For more general kernel K, as stated in [16], the spatial propagation mode of (1.9) with non-zero compactly supported initial function depends on the decaying property of K at $x = \pm\infty$: (i) when $K \in L^p(\mathbb{R})$ with $1 < p < \infty$ and $K(x) \geqslant (1+x)^{-\alpha}$ with $\alpha \in (0,1)$, the position of the "front" is of order $O(t^p)$; (ii) when $K \in L^\infty(\mathbb{R})$ and $K(+\infty) > 0$, the position of the "front" is of order $O(e^{\eta t})$ for some $\eta > 0$; (iii) when $K \in L^1(\mathbb{R})$ and $K = \bar{K}^{-1}$ for some kernel $\bar{K} \in W^{1,1}(\mathbb{R})$, only explicit upper and lower bounds on the spreading speed were obtained. Hence in the special case that $K(x)$ satisfies (1.10), the results of [50] presented a more refined dynamics by finding the precise spreading speed.

As recalled above, we find that if u_0 is a slowly decaying initial function, there is no result about the spatial propagation dynamics of (1.3) with $\chi \neq 0$, and the question whether or not the acceleration propagation occurs and how the chemotaxis affects (speeds up or slows down) the spatial propagation dynamics of (1.3) is unknown. We explore these questions in this paper and our main results are stated in the following theorem.

Theorem 1.1. *Assume $b > 2\chi\mu$ and u_0 satisfies (1.4) and (1.6). Then the following results hold.*

(i) *The solution of (1.3) satisfies*

$$\forall\, t \geqslant 0,\ \lim_{x\to+\infty} u(x,t) = 0,\ \textit{and}\ \liminf_{x\to-\infty} u(x,t) \to a/b \ \textit{as}\ t \to +\infty.$$

For any $\omega \in (0, a/b)$, there exists $T_\omega \geqslant 0$ such that the set $E_\omega(t)$ with $t \geqslant T_\omega$ is compact and non-empty, where $E_\omega(t) = \{x \in \mathbb{R}, u(x,t) = \omega\}$.

(ii) *For any $\epsilon \in (0, a)$, $\gamma_1 > 0$, and $\gamma_2 > 0$, if $\zeta(t)$ and $\eta(t)$ satisfy*

$$u_0(\zeta(t)) = \gamma_1 e^{-(a+\epsilon)t},\ u_0(\eta(t)) = \gamma_2 e^{-(a-\epsilon)t} \quad \textit{for } t \textit{ large enough},$$

then for any $\omega \in (0, a/b)$, there is a constant $T \geqslant T_\omega$ such that

$$E_\omega(t) \subseteq [\eta(t), \zeta(t)] \ \textit{for all}\ t \geqslant T.$$

(iii) *For any $\omega \in (0, a/b)$, we have that*

$$\lim_{t\to+\infty} \frac{\inf\{E_\omega(t)\}}{t} = +\infty.$$

Main proof ideas. The main difficulty of proving Theorem 1.1 lies in the failure of comparison principle of (1.3) due to the presence of chemotaxis which generates a cross diffusion. To overcome this obstacle, for any $\epsilon \in (0, a)$ same as in Theorem 1.1(ii), we construct the following two auxiliary equations for which the comparison principle holds (see Lemma 2.4):

$$w_t = w_{xx} + \frac{\chi\mu L}{\sqrt{\lambda}}|w_x| + w(a - (b - \chi\mu)w), \tag{1.11}$$

$$w_t = w_{xx} - \frac{\chi\mu L}{\sqrt{\lambda}}|w_x| + (a - \epsilon/2)w - Mw^{1+\delta}, \tag{1.12}$$

where L, M, and δ are some appropriate positive constants. By some estimates, we find that the solution of (1.3) is a lower solution of (1.11) but an upper solution of (1.12). Then by constructing an upper solution $\bar{w}$ of (1.11) and a lower solution $\underline{w}$ of (1.12), we can employ comparison principles to (1.11) and (1.12) and conclude that $\underline{w} \leqslant u \leqslant \bar{w}$. This is the primary new idea of this paper to overcome the lack of comparison principle in (1.3) and hence to prove Theorem 1.1.

Remark 1.2. For any $\epsilon \in (0, a)$, $\gamma_1, \gamma_2 > 0$, and $\omega \in (0, a/b)$, it follows from Theorem 1.1(ii) that

$$E_\omega(t) \subseteq [\eta(t), \zeta(t)] \subseteq u_0^{-1}\left\{\left[\gamma_1 e^{-(a+\epsilon)t}, \gamma_2 e^{-(a-\epsilon)t}\right]\right\} \text{ for } t \text{ large enough.} \tag{1.13}$$

Theorem 1.1(iii) shows that the average moving speed of the level set $E_\omega(t)$ tends to infinity as $t \to +\infty$, namely, the acceleration propagation occurs. Therefore, compared to [18], our results show that if chemotaxis is not strong (that is, $\chi < \frac{b}{2\mu}$), then it does not affect the spatial propagation dynamics of (1.3) with slowly decaying initial functions, in the sense that the estimates of upper and lower bounds of the acceleration propagation are the same as those for the Fisher-KPP equation (1.7) (see (1.8)). However, if chemotaxis strength is strong (that is, $\chi \geqslant \frac{b}{2\mu}$), it is unknown whether it has any effect on the spatial propagation dynamics of (1.3).

The rest of this paper is organized as follows. In Section 2, we present some preliminary results including the global existence of the classical solution of (1.3), some properties of initial function, and the comparison principle of the auxiliary equations (1.11) and (1.12). In Section 3, we prove Theorem 1.1 by estimating the upper bound and the lower bound of the moving level set.

2 | PREPARATORY RESULTS

In this section, we present some preliminary results. We first state the results about the global existence and asymptotic behavior of classical solutions of (1.3).

Lemma 2.1 (Salako and Shen [48]). *If $b > \chi\mu$, for any non-negative initial value $u_0 \in C^b_{\text{unif}}(\mathbb{R})$, (1.3) has a unique non-negative global classical solution (u, v) which is uniformly bounded in time and satisfies $u \in C([0, \infty), C^b_{\text{unif}}(\mathbb{R})) \cap C^1((0, \infty), C^b_{\text{unif}}(\mathbb{R}))$. Moreover, if $b > 2\chi\mu$ and $\inf_{x\in\mathbb{R}} u_0(x) > 0$, then*

$$\left\|u(\cdot, t) - \frac{a}{b}\right\|_\infty + \left\|v(\cdot, t) - \frac{a\mu}{b\lambda}\right\|_\infty \to 0 \text{ as } t \to +\infty. \tag{2.1}$$

The results of Lemma 2.1 directly indicate that: when $b > 2\chi\mu$, $(\phi, \psi) \equiv (\frac{a}{b}, \frac{a\mu}{b\lambda})$ is the unique solution of the following equations:

$$\begin{cases} \phi'' - \chi(\phi\psi')' + \phi(a - b\phi) = 0, & x \in \mathbb{R}, \\ \psi'' - \lambda\psi + \mu\phi = 0, & x \in \mathbb{R} \end{cases} \tag{2.2}$$

in $C^b_{\text{unif}}(\mathbb{R}) \times C^b_{\text{unif}}(\mathbb{R})$ satisfying $\inf_{x\in\mathbb{R}} \phi(x) > 0$, where prime means the differentiation with respect to x.

The following lemma states some properties of u_0 when it satisfies (1.4) and (1.6), which play crucial roles in the study of acceleration of propagation in the paper.

Lemma 2.2. *Assume that u_0 satisfies (1.4) and (1.6). We have the following statements:*

(i) $u_0'(x)/u_0(x) \to 0$ *as* $x \to +\infty$*;*

(ii) u_0 *decays more slowly than any exponentially decaying function as* $x \to +\infty$*, namely,*

$$\forall\, \kappa,\ \exists\, x_\kappa \text{ s.t. } u_0(x) \geqslant e^{-\kappa x} \ \text{ for all } \ x \in [x_\kappa, +\infty);$$

(iii) *for any* $\gamma_1 > 0$, $\gamma_2 > 0$, *and* $\rho_1 > \rho_2 > 0$, *if the functions* $y_1(\cdot)$ *and* $y_2(\cdot)$ *satisfy*

$$u_0(y_1(t)) = \gamma_1 e^{-\rho_1 t} \quad \text{and} \quad u_0(y_2(t)) = \gamma_2 e^{-\rho_2 t} \quad \text{for } t > 0 \text{ large enough},$$

then we have that

$$\lim_{t\to+\infty} (y_1(t) - y_2(t)) = +\infty.$$

Proof.

(i) Let $g(x) = u_0'(x)/u_0(x) \in C^1([\xi_0, +\infty))$. Then we obtain $g'(x) + g^2(x) = u_0''(x)/u_0(x)$, which implies

$$\lim_{x\to+\infty} (g'(x) + g^2(x)) = 0. \tag{2.3}$$

Note that $g(x) \leqslant 0$ for all $x \in [\xi_0, +\infty)$. In what follows, we prove that $\lim_{x\to+\infty} g(x)$ exists in $(-\infty, 0] \cup \{-\infty\}$. Otherwise, there are two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ satisfying that $y_n \to +\infty$, $z_n \to +\infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} g(y_n) = A,\ \lim_{n\to\infty} g(z_n) = B \text{ with } A, B \in (-\infty, 0] \cup \{-\infty\} \text{ and } A < B.$$

Let ε be a negative constant in (A, B). Since there exists $N_0 > 0$ such that $g(y_n) < \varepsilon$ and $g(z_n) > \varepsilon$ for any $n \geqslant N_0$, by the continuity of g and intermediate value theorem, there exits $x_n \in (\min\{y_n, z_n\}, \max\{y_n, z_n\})$ with $n \geqslant N_0$ such that $x_n \to +\infty$ as $n \to +\infty$, and

$$g(x_n) = \varepsilon \text{ for any } n \geqslant N_0. \tag{2.4}$$

Then by (2.3), we have that $\lim_{n\to+\infty} g'(x_n) = -\varepsilon^2 < 0$, which implies that there is $N_1 \geqslant N_0$ such that

$$g'(x_n) < -\varepsilon^2/2 < 0 \text{ for any } n \geqslant N_1. \tag{2.5}$$

Combining (2.4) with (2.5), by the continuity of g, we can find a sequence $\{\alpha_n\}$ with $n \geqslant N_1$ satisfying that

$$x_n \leqslant \alpha_n \leqslant x_{n+1},\ g(\alpha_n) < \varepsilon < 0,\ g'(\alpha_n) > 0 \text{ for any } n \geqslant N_1.$$

Then we have that $g'(\alpha_n) + g^2(\alpha_n) > \varepsilon^2$ for any $n \geqslant N_1$, which is a contradiction with (2.3). Therefore, $\lim_{x\to+\infty} g(x)$ exists in $(-\infty, 0] \cup \{-\infty\}$.

To complete the proof of (i), we suppose by contradiction that $\lim_{x\to+\infty} g(x) \neq 0$. Then by (2.3) we have that

$$\lim_{x\to+\infty} (g'(x) + g^2(x))/g^2(x) = 0.$$

Namely, $\lim_{x\to+\infty} g'(x)/g^2(x) = -1$. For any fixed constant $y \in [\xi_0, +\infty)$, it holds that

$$-\infty = \int_y^{+\infty} \frac{g'(x)}{g^2(x)} dx = \frac{1}{g(y)} - \frac{1}{g(+\infty)}$$

This is a contradiction since we suppose $g(+\infty) \neq 0$. Therefore, we have that $\lim_{x\to+\infty} g(x) = 0$.

(ii) By Lemma 2.2(i), we have that $u_0'(x)/u_0(x) \to 0$ as $x \to +\infty$. For any $\kappa > 0$, we can choose a constant η in $(0, \kappa)$. Then there exists $x_\eta > 0$ such that $u_0'(x)/u_0(x) \geqslant -\eta$ for $x \in [x_\eta, +\infty)$. By integrating this inequality from x_η to x, we can get that $u_0(x)/u_0(x_\eta) \geqslant e^{-\eta(x-x_\eta)}$ for any $x \in [x_\eta, +\infty)$. Hence, there exists $C_\eta = u_0(x_\eta)e^{\eta x_\eta} > 0$ such that

$$u_0(x) \geqslant C_\eta e^{-\eta x} \text{ for all } x \in [x_\eta, +\infty).$$

For any κ, we choose $x_\kappa = \max\{x_\eta, (\eta - \kappa)^{-1} \ln C_\eta\}$. It follows that

$$u_0(x) \geqslant C_\eta e^{(\kappa-\eta)x} e^{-\kappa x} \geqslant e^{-\kappa x} \ \text{ for all } \ x \in [x_\kappa, +\infty).$$

(iii) It suffices to prove that for any $M > 0$, there exists $t_M > 0$ such that

$$y_1(t) - y_2(t) \geqslant M \text{ for all } t \geqslant t_M.$$

By (1.4), (1.6), and $\rho_1 > \rho_2 > 0$, we can find a large constant $t_0 \geqslant 0$ such that the function $t \mapsto y_i(t)$ with $i \in \{1, 2\}$ is non-decreasing on $[t_0, +\infty)$ and

$$y_1(t) > y_2(t) \text{ for } t \geqslant t_0,\ y_2(t) \to +\infty \text{ as } t \to +\infty.$$

By $\rho_1 > \rho_2 > 0$ and Lemma 2.2(i), for any $M > 0$, there is a large constant $t_M \geqslant t_0$ such that

$$\frac{\gamma_1}{\gamma_2} e^{(\rho_2-\rho_1)t} \leqslant \frac{1}{2} \text{ for all } t \geqslant t_M, \tag{2.6}$$

and $y_2(t_M)$ is large enough such that

$$y_2(t_M) \geqslant \xi_0,\ \text{and } u_0'(x) \geqslant -\frac{1}{2M} u_0(x) \text{ for all } x \geqslant y_2(t_M).$$

When $t \geqslant t_M$, it follow from $y_2(t) \geqslant y_2(t_M)$ that

$$u_0'(x) \geqslant -\frac{1}{2M} u_0(x) \text{ for all } x \geqslant y_2(t).$$

Let t_M be larger (if necessary) such that

$$u_0(y_1(t)) = \gamma_1 e^{-\rho_1 t} \text{ and } u_0(y_2(t)) = \gamma_2 e^{-\rho_2 t} \text{ for all } t \geqslant t_M.$$

Then some calculations show that

$$\gamma_1 e^{-\rho_1 t} - \gamma_2 e^{-\rho_2 t} = u_0(y_1(t)) - u_0(y_2(t)) = \int_{y_2(t)}^{y_1(t)} u_0'(x)dx \geqslant -\frac{1}{2M}\int_{y_2(t)}^{y_1(t)} u_0(x)dx, \quad t \geqslant t_M.$$

Since u_0 is non-increasing on $[\xi_0, +\infty)$ and $y_2(t) - \geqslant y_2(t_M) \geqslant \xi_0$, we can get $u_0(x) \leqslant u_0(y_2(t))$ for $x \geqslant y_2(t)$, which implies that

$$\gamma_1 e^{-\rho_1 t} - \gamma_2 e^{-\rho_2 t} \geqslant -\frac{1}{2M}(y_1(t) - y_2(t))u_0(y_2(t)) = -\frac{1}{2M}(y_1(t) - y_2(t))\gamma_2 e^{-\rho_2 t}, \quad t \geqslant t_M.$$

By (2.6), we have that

$$y_1(t) - y_2(t) \geqslant -2M\left(\frac{\gamma_1}{\gamma_2} e^{(\rho_2 - \rho_1)t} - 1\right) \geqslant M, \quad t \geqslant t_M.$$

This completes the proof. □

Remark 2.3. We remark that the proof of Lemma 2.2(i) was given in [18] but hard to understand. Here we provide a different and easier proof. Lemma 2.2(ii) was announced in [18] without proof. Hence here we supplement a proof.

Let L be a positive constant and assume that $g : [0, L] \to \mathbb{R}$ is a Lipschitz continuous function. For $\alpha \in \mathbb{R}$, we consider the equation

$$\begin{cases} w_t = w_{xx} + \alpha |w_x| + g(w), & x \in \mathbb{R},\ t > 0, \\ w(x, 0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.7}$$

The following lemma provides a comparison principle for (2.7), which is a crucial tool used in the sequel.

Lemma 2.4 (Comparison principle). *Suppose that the bounded non-negative functions $\bar{w}$ and $\underline{w}$ are the upper and lower solutions of* (2.7)*, in the sense that*

$$\bar{w}_t - \bar{w}_{xx} - \alpha|\bar{w}_x| - g(\bar{w}) \geqslant 0 \geqslant \underline{w}_t - \underline{w}_{xx} - \alpha|\underline{w}_x| - g(\underline{w}), \quad \textit{for } x \in \mathbb{R},\ t > 0.$$

If $\bar{w}(x, 0) \geqslant \underline{w}(x, 0)$ for all $x \in \mathbb{R}$, then $\bar{w}(x, t) \geqslant \underline{w}(x, t)$ for all $(x, t) \in \mathbb{R} \times [0, +\infty)$.

Proof. Let $m(x, t) = \bar{w}(x, t) - \underline{w}(x, t)$ for $(x, t) \in \mathbb{R} \times [0, +\infty)$. We have that $m(x, 0) \geqslant 0$ for $x \in \mathbb{R}$ and $m_x(x, t) = \bar{w}_x(x, t) - \underline{w}_x(x, t)$. It follows that

$$-|m_x(x, t)| \leqslant |\bar{w}_x(x, t)| - |\underline{w}_x(x, t)| \leqslant |m_x(x, t)| \text{ for } (x, t) \in \mathbb{R} \times [0, +\infty).$$

Let $G > 0$ be a Lipschitz constant of g. It follows that

$$\begin{aligned} m_t(x,t) &\geqslant m_{xx}(x,t) + \alpha|\bar{w}_x(x,t)| - \alpha|\underline{w}_x(x,t)| + g(\bar{w}(x,t)) - g(\underline{w}(x,t)) \\ &\geqslant m_{xx}(x,t) - |\alpha m_x(x,t)| - G|m(x,t)| \quad \text{for } (x,t) \in \mathbb{R}\times[0,+\infty) \end{aligned} \tag{2.8}$$

Then it remains to prove that $m(x,t) \geqslant 0$ for all $(x,t) \in \mathbb{R}\times[0,+\infty)$. Suppose, by contradiction, that there exists $t_0 \in (0,+\infty)$ such that

$$\inf_{x\in\mathbb{R}}\{m(x,t_0)\} < 0.$$

Let $p(\cdot) : [0,1] \to [1,3]$ be a smooth increasing auxiliary function satisfying that

$$p(0) = 1,\ p(1) = 3,\ \text{and } p'(0) = p''(0) = p'(1) = p''(1) = 0. \tag{2.9}$$

We denote

$$C_1 = \max_{[0,1]}\{|p'(x)|\} \quad \text{and} \quad C_2 = \max_{[0,1]}\{|p''(x)|\}. \tag{2.10}$$

Let K be a large constant satisfying

$$K > \frac{2}{3}C_2 + \frac{2}{3}|\alpha|C_1 + \frac{8}{3}G. \tag{2.11}$$

Define

$$h(t) = -e^{-Kt}\inf_{x\in\mathbb{R}}\{m(x,t)\}, \quad t\in[0,t_0].$$

It follows that $h(0) \leqslant 0$ and $h(t_0) > 0$. Let $T_1 \in (0,t_0]$ satisfy

$$h(T_1) = H_0 \triangleq \max_{t\in[0,t_0]}\{h(t)\} > 0.$$

Then we get

$$\inf_{x\in\mathbb{R}}\{m(x,T_1)\} = -H_0e^{KT_1}, \tag{2.12}$$

and

$$m(x,t) \geqslant \inf_{x\in\mathbb{R}}\{m(x,t)\} = -h(t)e^{Kt} \geqslant -H_0e^{Kt} \text{ for all } x\in\mathbb{R},\ t\in[0,t_0]. \tag{2.13}$$

Now we consider the following two cases

Case A There exists $x_0 \in \mathbb{R}$ such that $m(x_0,T_1) = \inf_{x\in\mathbb{R}}\{m(x,T_1)\}$.
Case B $\liminf_{x\to+\infty}\{m(x,T_1)\} = \inf_{x\in\mathbb{R}}\{m(x,T_1)\}$ or $\liminf_{x\to-\infty}\{w(x,T_1)\} = \inf_{x\in\mathbb{R}}\{m(x,T_1)\}$.

If case A holds, we can obtain that $m(x_0,T_1) = -H_0e^{KT_1}$, $m_x(x_0,T_1) = 0$, and $m_{xx}(x_0,T_1) \geqslant 0$. By (2.12) and (2.13), we have that

$$m(x_0,t) + H_0e^{Kt} \geqslant 0 = m(x_0,T_1) + H_0e^{KT_1} \text{ for } t\in[0,T_1].$$

It follows that

$$0 \geqslant \frac{d}{dt}\left(m(x_0,t)+H_0e^{Kt}\right)\Big|_{t=T_1} = m_t(x_0,T_1)+KH_0e^{KT_1}. \tag{2.14}$$

Then we get by (2.11) that

$$m_t(x_0,T_1)-m_{xx}(x_0,T_1)+|\alpha m_x(x_0,T_1)|+G|m(x_0,T_1)| \leqslant (-K+G)H_0e^{KT_1} < 0,$$

which contradicts (2.8).

If case B holds, then there exists $x_1 \in \mathbb{R}$ (far away from 0) such that $m(x_1,T_1) < -\frac{3}{4}H_0e^{KT_1}$. We define a function $z(\cdot) : \mathbb{R} \to [1,3]$ as follows:

$$z(x) = \begin{cases} 1 & \text{for } |x| \leqslant |x_1|, \\ p(|x|-|x_1|) & \text{for } |x_1| < |x| < |x_1|+1, \\ 3 & \text{for } |x| \geqslant |x_1|+1. \end{cases}$$

It follows from (2.9) that $z(\cdot) \in C^2(\mathbb{R})$. For $\sigma > 0$, we define

$$\rho_\sigma(x,t) = -\left(\frac{1}{2}+\sigma z(x)\right)H_0e^{Kt}$$

and

$$\sigma^* = \inf\{\sigma > 0 \mid \rho_\sigma(x,t) \leqslant m(x,t) \text{ for all } x \in \mathbb{R},\ t \in [0,T_1]\}.$$

Then $\rho_{\frac{1}{4}}(x_1,T_1) = -\frac{3}{4}H_0e^{KT_1} > m(x_1,T_1)$. By (2.13), we have that $\rho_{\frac{1}{2}}(x,t) \leqslant -H_0e^{Kt} \leqslant m(x,t)$ for all $x \in \mathbb{R}$ and $t \in [0,T_1]$. The monotonicity of $\sigma \mapsto \rho_\sigma$ implies that $\frac{1}{4} \leqslant \sigma^* \leqslant \frac{1}{2}$. When $|x| \geqslant |x_1|+1$, by (2.13) we get that

$$\rho_{\sigma^*}(x,t) = -\left(\frac{1}{2}+3\sigma^*\right)H_0e^{Kt} \leqslant -\frac{5}{4}H_0e^{Kt} < m(x,t) \text{ for all } t \in [0,T_1].$$

By the definition of σ^*, there exist $T_2 \in (0,T_1]$ and $x_2 \in [-|x_1|-1, |x_1|+1]$ such that

$$m(x_2,T_2) = \rho_{\sigma^*}(x_2,T_2), \text{ and } m(x,t) \geqslant \rho_{\sigma^*}(x,t) \text{ for all } x \in \mathbb{R},\ t \in [0,T_1]. \tag{2.15}$$

From $\frac{1}{4} \leqslant \sigma^* \leqslant \frac{1}{2}$ and $1 \leqslant z(x) \leqslant 3$, it follows that

$$-2H_0e^{KT_2} \leqslant m(x_2,T_2) = \rho_{\sigma^*}(x_2,T_2) \leqslant -\frac{3}{4}H_0e^{KT_2}. \tag{2.16}$$

We have from (2.15) that

$$\frac{\partial}{\partial x}(m(x,T_2)-\rho_{\sigma^*}(x,T_2))\Big|_{x=x_2} = 0, \quad \text{and} \quad \frac{\partial^2}{\partial x^2}(m(x,T_2)-\rho_{\sigma^*}(x,T_2))\Big|_{x=x_2} \geqslant 0.$$

It follows from (2.10) and $\sigma^* \leqslant \frac{1}{2}$ that

$$|m_x(x_2, T_2)| = \left|\frac{\partial}{\partial x}\rho_{\sigma^*}(x_2, T_2)\right| = \sigma^*|z'(x_2)|H_0 e^{KT_2} \leqslant \frac{1}{2}C_1 H_0 e^{KT_2},$$

$$m_{xx}(x_2, T_2) \geqslant \frac{\partial^2}{\partial x^2}\rho_{\sigma^*}(x_2, T_2) = -\sigma^* z''(x_2)H_0 e^{KT_2} \geqslant -\frac{1}{2}C_2 H_0 e^{KT_2}.$$

Moreover, it follows from (2.15) that

$$\frac{\partial}{\partial t}(m(x_2, t) - \rho_{\sigma^*}(x_2, t))\Big|_{t=T_2} \leqslant 0,$$

which along with (2.16) implies that

$$m_t(x_2, T_2) \leqslant \frac{\partial}{\partial t}\rho_{\sigma^*}(x_2, T_2) = K\rho_{\sigma^*}(x_2, T_2) \leqslant -\frac{3}{4}KH_0 e^{KT_2}.$$

Therefore, we get from (2.11) that

$$m_t(x_2, T_2) - m_{xx}(x_2, T_2) + |\alpha m_x(x_2, T_2)| + G|m(x_2, T_2)|$$
$$\leqslant \left(-\frac{3}{4}K + \frac{1}{2}C_2 + \frac{1}{2}|\alpha|C_1 + 2G\right)H_0 e^{KT_2} < 0,$$

which contradicts (2.8). Thus the proof is completed. □

3 | ACCELERATION OF PROPAGATION

In this section, we are devoted to the proof of Theorem 1.1. By substituting the second equation of (1.3) into the first one, we get

$$u_t = u_{xx} - \chi u_x v_x + g(u, v), \ x \in \mathbb{R}, \ t > 0, \tag{3.1}$$

where

$$g(u, v) = u[a - bu + \chi\mu u - \chi\lambda v]. \tag{3.2}$$

The following lemma provides an upper bound of the moving level set $E_\omega(t)$ for large t.

Lemma 3.1. *Under the same assumptions as in Theorem 1.1, for any $\epsilon > 0$ and $\gamma_1 > 0$, if $\zeta(t)$ satisfies that*

$$u_0(\zeta(t)) = \gamma_1 e^{-(a+\epsilon)t} \text{ for } t \text{ large enough},$$

then for any $\omega \in (0, a/b)$, there is a constant $T_1 \geqslant 0$ such that

$$E_\omega(t) \subseteq (-\infty, \zeta(t)] \text{ for any } t \geqslant T_1. \tag{3.3}$$

Proof. By $u \in C([0,\infty), C^b_{\text{unif}}(\mathbb{R}))$ and (2.1), we can define

$$L = \sup_{t\in[0,+\infty)} \|u(\cdot, t)\|_\infty < +\infty, \tag{3.4}$$

which implies that

$$0 \leqslant u(x,t) \leqslant L \text{ for all } (x,t) \in \mathbb{R} \times [0,+\infty). \tag{3.5}$$

Note that for any given function u, the second equation of (1.3) can be regarded as an ordinary differential equation for v. As stated in [50, Lemma 2.1], its solution in $C^b_{\text{unif}}(\mathbb{R})$ is written as

$$v(x,t) = \frac{\mu}{\lambda} J * u(x,t) = \frac{\mu}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\sqrt{\lambda}|x-z|} u(z,t)dz \quad \text{with } J(x) = \frac{\sqrt{\lambda}}{2} e^{-\sqrt{\lambda}|x|}.$$

It follows from (3.5) and $\int_{\mathbb{R}} J(x)dx = 1$ that

$$0 \leqslant v(x,t) \leqslant \frac{\mu L}{\lambda} \quad \text{for all } x \in \mathbb{R},\ t > 0.$$

By [50, Lemma 2.2], we have $|v_x(x,t)| \leqslant \sqrt{\lambda} v(x,t)$, which along with the above inequality yields

$$|v_x(x,t)| \leqslant \frac{\mu L}{\sqrt{\lambda}} \quad \text{for all } x \in \mathbb{R},\ t > 0. \tag{3.6}$$

With (3.2), we have that $g(u,v) \leqslant u(a-(b-\chi\mu)u)$. It follows from (3.1) and (3.6) that

$$u_t = u_{xx} + \chi v_x u_x + g(u,v) \leqslant u_{xx} + \frac{\chi\mu L}{\sqrt{\lambda}} |u_x| + u(a-(b-\chi\mu)u) \quad \text{for all } x \in \mathbb{R},\ t > 0.$$

Then $u(x,t)$ is a lower solution of the following equation:

$$w_t = w_{xx} + \frac{\chi\mu L}{\sqrt{\lambda}} |w_x| + w(a-(b-\chi\mu)w), \qquad x \in \mathbb{R},\ t > 0. \tag{3.7}$$

By (1.4), (1.6), and Lemma 2.2(i), we have that

$$u_0(x) \to 0,\ u_0'(x)/u_0(x) \to 0, \text{ and } u_0''(x)/u_0(x) \to 0 \text{ as } x \to +\infty. \tag{3.8}$$

Let ξ_0 be the constant in (1.6). Then it follows from (1.4) that $\inf_{x\in(-\infty,\xi_0]}\{u_0(x)\} > 0$. Then for any $\epsilon > 0$, we can choose a constant $\xi_1 \in [\xi_0,+\infty)$ such that

$$|u_0''(x)| \leqslant \frac{\epsilon}{4} u_0(x), \qquad \frac{\chi\mu L}{\sqrt{\lambda}} |u_0'(x)| \leqslant \frac{\epsilon}{4} u_0(x) \quad \text{for all } x \geqslant \xi_1, \tag{3.9}$$

and

$$0 < u_0(\xi_1) < \inf_{x\in(-\infty,\xi_0]} \{u_0(x)\}. \tag{3.10}$$

We define a function $\theta(\cdot) : [0,+\infty) \to (0,+\infty)$ satisfying

$$\begin{cases} \theta'(t) = \theta(t)(a-(b-\chi\mu)\theta(t)), & t > 0, \\ \theta(0) = \theta_0, \end{cases}$$

where

$$\theta_0 = \max\left\{\sup_{x\in\mathbb{R}}\{u_0(x)\}, \frac{a}{b-\chi\mu}\right\}.$$

Note that $\theta(\cdot)$ is non-increasing, and if $\sup_{x\in\mathbb{R}}\{u_0(x)\} \leqslant a/(b-\chi\mu)$, then $\theta(t) \equiv a/(b-\chi\mu)$. Define

$$\bar{w}(x,t) = \min\left\{Cu_0(x)e^{\rho t}, \theta(t)\right\} \text{ for } x \in \mathbb{R},\ t \geqslant 0, \tag{3.11}$$

where

$$\rho = a + \epsilon/2 \quad \text{and} \quad C = \frac{\theta_0}{u_0(\xi_1)} \geqslant \frac{\sup_{x\in\mathbb{R}}\{u_0(x)\}}{u_0(\xi_1)} \geqslant 1.$$

Now we show that $\bar{w}(x,t)$ is an upper solution of (3.7). When $\bar{w}(x,t) = \theta(t)$, we easily check

$$\bar{w}_t - \bar{w}_{xx} - \frac{\chi\mu L}{\sqrt{\lambda}}|\bar{w}_x| - \bar{w}(a-(b-\chi\mu)\bar{w}) = \theta'(t) - \theta(t)(a-(b-\chi\mu)\theta(t)) = 0.$$

When $\bar{w}(x,t) = Cu_0(x)e^{\rho t}$, we have that

$$Cu_0(x) \leqslant Cu_0(x)e^{\rho t} \leqslant \theta(t) \leqslant \theta_0,$$

which implies that $u_0(x) \leqslant \theta_0/C = u_0(\xi_1)$. By (3.10) and $u_0' \leqslant 0$ on $[\xi_0, +\infty)$, we obtain $x \geqslant \xi_1$. From (3.9) and $\bar{w}(x,t) = Cu_0(x)e^{\rho t}$, it follows that

$$\begin{aligned}
&\bar{w}_t - \bar{w}_{xx} - \frac{\chi\mu L}{\sqrt{\lambda}}|\bar{w}_x| - \bar{w}(a-(b-\chi\mu)\bar{w}) \\
&\geqslant \rho Cu_0(x)e^{\rho t} - Cu_0''(x)e^{\rho t} - \frac{\chi\mu L}{\sqrt{\lambda}}C|u_0'(x)|e^{\rho t} - aCu_0(x)e^{\rho t} \\
&\geqslant Cu_0(x)e^{\rho t}\left(\rho - \frac{\epsilon}{4} - \frac{\epsilon}{4} - a\right) = 0.
\end{aligned}$$

Therefore, $\bar{w}(x,t)$ is an upper solution of (3.7).

By $C \geqslant 1$ and $\theta_0 \geqslant \sup_{x\in\mathbb{R}}\{u_0(x)\}$, we get that

$$\bar{w}(x,0) = \min\{Cu_0(x), \theta_0\} \geqslant u_0(x) \text{ for all } x \in \mathbb{R}.$$

Applying Lemma 2.4 to (3.7), we have that

$$u(x,t) \leqslant \bar{w}(x,t) \text{ for all } x \in \mathbb{R},\ t \geqslant 0. \tag{3.12}$$

For $\omega \in (0, a/b)$ and $t \geqslant 0$, when $E_\omega(t)$ is an empty set, (3.3) is obvious. When $E_\omega(t)$ is non-empty, for any $x_\omega(t) \in E_\omega(t)$, we get from (3.12) that

$$\omega = u(x_\omega(t), t) \leqslant \bar{w}(x_\omega(t), t) \leqslant Cu_0(x_\omega(t))e^{\rho t}. \tag{3.13}$$

Since $\zeta(\cdot)$ satisfies

$$u_0(\zeta(t)) = \gamma_1 e^{-(a+\epsilon)t} \text{ for } t \text{ large enough,}$$

we can get from (1.4) and (1.6) that $\zeta(t) \to +\infty$ as $t \to +\infty$. Let T_1 be a constant large enough such that $\zeta(t) \geqslant \xi_0$, $u_0(\zeta(t)) = \gamma_1 e^{-(a+\epsilon)t}$ for all $t \geqslant T_1$, and $T_1 > \max\{0, 2\epsilon^{-1}\ln(\gamma_1 C/\omega)\}$. By (3.13) and $\rho = a + \epsilon/2$, we get that

$$u_0(x_\omega(t)) \geqslant C^{-1}\omega e^{-\rho t} \geqslant C^{-1}\omega e^{\epsilon T_1/2} e^{-(a+\epsilon)t} > \gamma_1 e^{-(a+\epsilon)t} = u_0(\zeta(t)) \quad \text{for all } t \geqslant T_1.$$

This implies $x_\omega(t) \leqslant \zeta(t)$ for all $t \geqslant T_1$ from (1.6). Hence, we obtain (3.3) and complete the proof. □

In the following lemma, we give the lower bound of the moving level set $E_\omega(t)$ for large t.

Lemma 3.2. *Under the same assumptions as in Theorem 1.1, for any $\epsilon \in (0, a)$ and $\gamma_2 > 0$, if $\eta(t)$ satisfies*

$$u_0(\eta(t)) = \gamma_2 e^{-(a-\epsilon)t} \quad \textit{for } t \textit{ large enough},$$

then for any $\omega \in (0, a/b)$, there is a constant $T_2 \geqslant 0$ such that

$$E_\omega(t) \subseteq [\eta(t), +\infty) \textit{ for any } t \geqslant T_2. \tag{3.14}$$

Proof. By [50, Lemma 2.5 and (2.21)], for any $r \gg 1$ and $p > 1$, there exist $C_{r,p} > 0$ and $\varepsilon_r > 0$ such that

$$\chi\lambda v(x,t) \leqslant C_{r,p}(u(x,t))^{\frac{1}{p}} + q\varepsilon_r \qquad \text{for all } x \in \mathbb{R},\ t > 0, \tag{3.15}$$

where $q \triangleq \max\{\|u_0\|_\infty, a/(b-\chi\mu)\}$ and the constant ε_r satisfies $\lim_{r\to\infty} \varepsilon_r = 0$. Fix $\epsilon \in (0, a)$ and choose r large enough such that

$$q\varepsilon_r < \epsilon/2.$$

Denote

$$\delta = 1/p \quad \text{and} \quad M = C_{r,p} + (b-\chi\mu)L^{1-\delta},$$

where L is defined by (3.4). From (3.2), (3.5), and (3.15), it follows that

$$\begin{aligned} g(u,v) &= u[a - (b-\chi\mu)u - \chi\lambda v] \\ &\geqslant u[a - (b-\chi\mu)L^{1-\delta}u^\delta - C_{r,p}u^\delta - q\varepsilon_r] \\ &\geqslant (a-\epsilon/2)u - Mu^{1+\delta}. \end{aligned}$$

We get from (3.1) and (3.6) that

$$u_t = u_{xx} - \chi v_x u_x + g(u,v) \geqslant u_{xx} - \frac{\chi\mu L}{\sqrt{\lambda}}|u_x| + (a-\epsilon/2)u - Mu^{1+\delta} \quad \text{for all } x \in \mathbb{R},\ t > 0.$$

Then $u(x,t)$ is an upper solution of the following equation:

$$w_t = w_{xx} - \frac{\chi\mu L}{\sqrt{\lambda}}|w_x| + (a-\epsilon/2)w - Mw^{1+\delta}, \quad x \in \mathbb{R},\ t > 0. \tag{3.16}$$

Choose a constant $\rho > 0$ satisfying that

$$\max\left\{a-\epsilon, \frac{a-\epsilon/2}{1+\delta}\right\} < \rho < a-\epsilon/2.$$

By (3.8) and (1.4), we can find a constant $\xi_2 \in [\xi_0, +\infty)$ large enough such that

$$G_1(x) \triangleq \frac{|u_0''(x)|}{u_0(x)} + \frac{\chi\mu L}{\sqrt{\lambda}}\frac{|u_0'(x)|}{u_0(x)} \leqslant (a-\epsilon/2)-\rho, \tag{3.17}$$

$$\begin{aligned} G_2(x) &\triangleq (1+\delta)\left[\frac{|u_0''(x)|}{u_0(x)} + \delta\left(\frac{u_0'(x)}{u_0(x)}\right)^2\right] + \frac{\chi\mu L}{\sqrt{\lambda}}(1+\delta)\frac{|u_0'(x)|}{u_0(x)} \\ &\leqslant \frac{\rho(1+\delta)-(a-\epsilon/2)}{2} \end{aligned} \tag{3.18}$$

for all $x \geqslant \xi_2$, and

$$0 < u_0(\xi_2) < \inf_{x\in(-\infty,\xi_0]}\{u_0(x)\}. \tag{3.19}$$

We define a function

$$F(s) = s - Bs^{1+\delta},\ s \geqslant 0,$$

where

$$B = \max\left\{\frac{1}{u_0^{\delta}(\xi_2)}, \frac{2M}{\rho(1+\delta)-(a-\epsilon/2)}\right\}. \tag{3.20}$$

Denote $s_0 \triangleq B^{-1/\delta} \leqslant u_0(\xi_2)$ and $s_1 \triangleq (1+\delta)^{-1/\delta}B^{-1/\delta} < s_0$. Then we have that

$$F(s) \leqslant 0 \text{ for all } s \geqslant s_0, \tag{3.21}$$

and

$$F(s) \leqslant F_0 \triangleq \max_{s\geqslant 0}\{F(s)\} = F(s_1) = \frac{\delta B^{-1/\delta}}{(1+\delta)^{1+1/\delta}} \leqslant B^{-1/\delta} \leqslant u_0(\xi_2) \text{ for all } s \geqslant 0. \tag{3.22}$$

Next we prove that the function defined by

$$\underline{w}(x,t) = \max\left\{F(u_0(x)e^{\rho t}), 0\right\} = \max\left\{u_0(x)e^{\rho t} - Bu_0^{1+\delta}(x)e^{\rho(1+\delta)t}, 0\right\},\ x \in \mathbb{R},\ t \geqslant 0$$

is a lower solution of (3.16). It suffices to check that $\underline{w}$ is a lower solution in the region where $\underline{w} > 0$. From (3.21), it follows that $u_0(x)e^{\rho t} < s_0 \leqslant u_0(\xi_2)$, which implies $u_0(x) < u_0(\xi_2)$. By (3.19) and $u_0'(x) \leqslant 0$ on $[\xi_0, +\infty)$, we have that

$$x > \xi_2, \text{ when } \underline{w}(x,t) > 0. \tag{3.23}$$

Note that

$$\left|\underline{w}_x(x,t)\right| \leqslant H_1(x,t) \triangleq |u_0'(x)|e^{\rho t} + B(1+\delta)u_0^{\delta}(x)|u_0'(x)|e^{\rho(1+\delta)t},$$

$$\left|\underline{w}_{xx}(x,t)\right| \leqslant H_2(x,t) \triangleq |u_0''(x)|e^{\rho t} + B(1+\delta)\left[u_0^{\delta}(x)|u_0''(x)| + \delta u_0^{\delta-1}(x)(u_0'(x))^2\right]e^{\rho(1+\delta)t}.$$

Some calculations show that

$$\begin{aligned}
&\underline{w}_t(x,t) - \underline{w}_{xx}(x,t) + \frac{\chi\mu L}{\sqrt{\lambda}}\left|\underline{w}_x(x,t)\right| - (a-\epsilon/2)\underline{w}(x,t) + M\underline{w}^{1+\delta}(x,t)\\
&\leqslant \rho u_0(x)e^{\rho t} - B\rho(1+\delta)u_0^{1+\delta}(x)e^{\rho(1+\delta)t} + H_2(x,t) + \frac{\chi\mu L}{\sqrt{\lambda}}H_1(x,t)\\
&\quad - (a-\epsilon/2)u_0(x)e^{\rho t} + (a-\epsilon/2)Bu_0^{1+\delta}(x)e^{\rho(1+\delta)t} + Mu_0^{1+\delta}(x)e^{\rho(1+\delta)t}\\
&= u_0(x)e^{\rho t}[\rho + G_1(x) - (a-\epsilon/2)] + Bu_0^{1+\delta}(x)e^{\rho(1+\delta)t}[-\rho(1+\delta) + G_2(x) + (a-\epsilon/2) + M/B].
\end{aligned}$$

By (3.17), (3.18), and (3.20), we obtain that

$$\rho + G_1(x) - (a-\epsilon/2) \leqslant 0, \text{ and } -\rho(1+\delta) + G_2(x) + (a-\epsilon/2) + M/B \leqslant 0.$$

Then we have that

$$\underline{w}_t(x,t) - \underline{w}_{xx}(x,t) + \frac{\chi\mu L}{\sqrt{\lambda}}\left|\underline{w}_x(x,t)\right| - (a-\epsilon/2)\underline{w}(x,t) + M\underline{w}^{1+\delta}(x,t) \leqslant 0.$$

Therefore, $\underline{w}(x,t)$ is a lower solution of (3.16).

Note that $s_1e^{-\rho t} \leqslant s_1 < u_0(\xi_2)$ for all $t \geqslant 0$. By (1.4) and the non-increasing property of $u_0(\cdot)$ on $[\xi_2, +\infty)$, there exists $z(t) \geqslant \xi_2$ for any $t \geqslant 0$ satisfying $\lim_{t\to\infty} z(t) = +\infty$ and

$$u_0(z(t)) = s_1e^{-\rho t},\ t \geqslant 0. \tag{3.24}$$

It follows that

$$\underline{w}(z(t),t) = \max\{F(s_1), 0\} = F_0 \text{ for all } t \geqslant 0. \tag{3.25}$$

Next we prove that

$$u(x,t) \geqslant F_0 \text{ for all } x \leqslant z(t),\ t \geqslant 0. \tag{3.26}$$

Consider the following function:

$$w_0(x) = \begin{cases} u_0(\xi_2), & \text{for } x \leqslant \xi_2,\\ u_0(x), & \text{for } x \geqslant \xi_2.\end{cases}$$

Note that $w_0(\cdot)$ is a non-increasing function, namely, $w_0(x+y) \leqslant w_0(x)$ for any $x \in \mathbb{R}$ and $y \geqslant 0$. By (3.19) and the non-increasing property of $u_0(\cdot)$ on $[\xi_0, +\infty)$, we get $w_0(x) \leqslant u_0(x)$ for $x \in \mathbb{R}$. With (3.22), we can easily check that

$$\underline{w}(x,0) = \max\{u_0(x) - Bu_0^{1+\delta}(x), 0\} \leqslant \min\{u_0(x), F_0\} \leqslant \min\{u_0(x), u_0(\xi_2)\} = w_0(x),\ x \in \mathbb{R}.$$

Then for any constant $y \geqslant 0$, we have that

$$\underline{w}(x+y,0) \leqslant w_0(x+y) \leqslant w_0(x) \leqslant u_0(x),\ x \in \mathbb{R}.$$

Since $\underline{w}(x,t)$ is a lower solution of (3.16), for any constant $y \geqslant 0$, $\underline{w}(x+y,t)$ is also a lower solution of (3.16). Note that $u(x,t)$ is an upper solution of (3.16). Applying Lemma 2.4 to (3.16), we have that

$$u(x,t) \geqslant \underline{w}(x+y,t) \text{ for all } x \in \mathbb{R},\ t \geqslant 0,\ y \geqslant 0.$$

For any $t \geqslant 0$ and $x \leqslant z(t)$, we can choose $y = z(t) - x \geqslant 0$, and it holds by (3.25) that

$$u(x,t) \geqslant \underline{w}(x+y,t) = \underline{w}(z(t),t) = F_0.$$

Then we obtain (3.26), which implies that

$$\liminf_{t\to+\infty} \inf_{x\leqslant z(t)} u(x,t) \geqslant F_0. \tag{3.27}$$

Recall that for any $\epsilon \in (0,a)$ and $\gamma_2 > 0$, the function $\eta(\cdot)$ satisfies

$$u_0(\eta(t)) = \gamma_2 e^{-(a-\epsilon)t} \quad \text{for } t \text{ large enough.} \tag{3.28}$$

Next we prove that

$$\lim_{t\to+\infty} \sup_{x\leqslant\eta(t)} \left| u(x,t) - \frac{a}{b} \right| = 0. \tag{3.29}$$

Suppose by contradiction that (3.29) does not hold. Then there exist two sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ satisfying $t_n \to +\infty$, $x_n \leqslant \eta(t_n)$ for n large enough, and

$$\liminf_{n\to\infty} \left| u(x_n,t_n) - \frac{a}{b} \right| > 0. \tag{3.30}$$

We consider the function sequence $(u^n(x,t), v^n(x,t)) = (u(x+x_n, t+t_n), v(x+x_n, t+t_n))$. By some estimates similar to those in parabolic equations, we have that $(u_n(x,t), v_n(x,t))$ converges, up to extraction of a subsequence, to some function $(u^*(x,t), v^*(x,t))$ locally uniformly in $C^{2,1}(\mathbb{R}\times\mathbb{R}) \times C^{2,1}(\mathbb{R}\times\mathbb{R})$. It follows that $(u^*(x,t), v^*(x,t))$ is an entire solution (which means that it is defined for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$) of the following system:

$$\begin{cases} u_t = u_{xx} - \chi(uv_x)_x + u(a-bu), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ 0 = v_{xx} - \lambda v + \mu u, & x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases} \tag{3.31}$$

We claim that

$$u^*(x,t) \geqslant F_0 \quad \text{for all } \ x \in \mathbb{R},\ t \in \mathbb{R}. \tag{3.32}$$

By $x_n \leqslant \eta(t_n)$ and $u^*(x,t) = \lim_{n\to\infty} u(x+x_n, t+t_n)$, it holds that

$$u^*(x,t) \geqslant \liminf_{\tau\to+\infty} \inf_{y\leqslant\eta(\tau)} u(x+y, t+\tau) = \liminf_{\tau\to+\infty} \inf_{y\leqslant x+\eta(\tau-t)} u(y,\tau), \ \ x \in \mathbb{R},\ t \in \mathbb{R}. \tag{3.33}$$

By (3.24) and (3.28), we have that

$$u_0(z(\tau)) = s_1 e^{-\rho\tau}, \text{ and } u_0(\eta(\tau - t)) = \gamma_2 e^{(a-\epsilon)t} e^{-(a-\epsilon)\tau} \quad \text{for } \tau \text{ large enough.}$$

For any fixed $t \in \mathbb{R}$, by $\rho > a - \epsilon > 0$ and Lemma 2.2(iii), we can get that

$$\lim_{\tau\to+\infty} z(\tau) - \eta(\tau - t) = +\infty.$$

Then for any fixed $x \in \mathbb{R}$ and $t \in \mathbb{R}$, we have that

$$z(\tau) \geqslant x + \eta(\tau - t) \quad \text{for } \tau \text{ large enough.}$$

It follows from (3.27) and (3.33) that

$$u^*(x,t) \geqslant \liminf_{\tau\to+\infty} \inf_{y\leqslant z(\tau)} u(y,\tau) \geqslant F_0 \text{ for all } \ x \in \mathbb{R},\ t \in \mathbb{R},$$

which means that (3.32) holds. Note that both $(u^*(x,-\infty), v^*(x,-\infty))$ and $(u^*(x,+\infty), v^*(x,+\infty))$ are the solution of (2.2) in $C^b_{\text{unif}}(\mathbb{R}) \times C^b_{\text{unif}}(\mathbb{R})$ satisfying

$$\inf_{x\in\mathbb{R}} u^*(x,-\infty) \geqslant F_0 > 0, \text{ and } \inf_{x\in\mathbb{R}} u^*(x,+\infty) \geqslant F_0 > 0.$$

When $b > 2\chi\mu$, Lemma 2.1 implies that $(u^*(\cdot,\pm\infty), v^*(\cdot,\pm\infty)) \equiv (\frac{a}{b}, \frac{a\mu}{b\lambda})$. Then by the stability of the positive constant equilibrium $(\frac{a}{b}, \frac{a\mu}{b\lambda})$, it holds that $u^*(x,t) \equiv a/b$. In particular, we have

$$\frac{a}{b} = u^*(0,0) = \lim_{n\to\infty} u(x_n, t_n),$$

which contradicts (3.30). Therefore, we obtain (3.29).

Finally, we complete the proof of Lemma 3.2 by (3.29), which implies that for any $\omega \in (0, a/b)$, there exists $T_2 \geqslant 0$ such that

$$\inf_{x\leqslant\eta(t)} u(x,t) > \omega \text{ for all } t \geqslant T_2. \tag{3.34}$$

For any $\omega \in (0, a/b)$, when $E_\omega(t)$ is an empty set, (3.14) is obvious. When $E_\omega(t)$ is non-empty, for any $x_\omega(t) \in E_\omega(t)$, we get from (3.34) that

$$x_\omega(t) \geqslant \eta(t) \text{ for all } t \geqslant T_2,$$

which implies (3.14) and completes the proof. □

Proof of Theorem 1.1. For any fixed $t \geqslant 0$, it follows from (3.11) and (3.12) that

$$\lim_{x\to+\infty} u(x,t) \leqslant \lim_{x\to+\infty} \bar{w}(x,t) \leqslant \lim_{x\to+\infty} Cu_0(x)e^{\rho t} = 0,$$

which implies $\lim\limits_{x\to+\infty} u(x,t) = 0$ due to the non-negativity of u. By (3.29) and $\eta(t) \to +\infty$ as $t \to +\infty$, we have that

$$\liminf_{x\to-\infty} u(x,t) \to a/b \text{ as } t \to +\infty.$$

Then for any $\omega \in (0, a/b)$, we can find $T_\omega \geqslant 0$ such that $\liminf\limits_{x\to-\infty} u(x,t) > \omega$ for all $t \geqslant T_\omega$. By the continuity of $u(\cdot,t)$, the set $E_\omega(t)$ with $t \geqslant T_\omega$ is compact and non-empty. Then the assertion of Theorem 1.1(i) is proved.

The result asserted in Theorem 1.1 (ii) is directly obtained from Lemmas 3.1 and 3.2 by denoting $T = \max\{T_1, T_2\}$.

Next we prove Theorem 1.1(iii) by Lemmas 2.2(ii) and 3.2. It suffices to show that for any $N > 0$, there exists a constant $T_N \geqslant 0$ such that

$$\frac{\inf\{E_\omega(t)\}}{t} \geqslant N \text{ for all } t \geqslant T_N. \tag{3.35}$$

Let $\gamma_2 > 1$ and choose $T_0 > 0$ large enough such that

$$u_0(\eta(t)) = \gamma_2 e^{-(a-\epsilon)t} \quad \text{for all } t \geqslant T_0.$$

For any $N > 0$, we define a constant κ by

$$\kappa = \frac{a - \epsilon - R\ln\gamma_2}{N},$$

where $R > 0$ is a constant small enough such that $\kappa > 0$. Let x_κ denote the constant given by Lemma 2.2(ii). Recall that $\eta(t) \to +\infty$ as $t \to +\infty$. Then there exists a constant $T_N \geqslant \max\{T_0, T_2, 1/R\}$ such that

$$\eta(t) \geqslant \max\{x_\kappa, \xi_0\} \quad \text{for all } t \geqslant T_N.$$

When $t \geqslant T_N \geqslant T_2$, for any $x_\omega(t) \in E_\omega(t)$, it follows from Lemma 3.2 that

$$\max\{x_\kappa, \xi_0\} \leqslant \eta(t) \leqslant x_\omega(t),$$

which, along with the non-increasing property of u_0 on $[\xi_0, +\infty)$, implies that

$$u_0(x_\omega(t)) \leqslant u_0(\eta(t)) = \gamma_2 e^{-(a-\epsilon)t},\ t \geqslant T_N.$$

By $x_\omega(t) \geqslant \eta(t) \geqslant x_\kappa$ for $t \geqslant T_N$, Lemma 2.2(ii) shows that

$$u_0(x_\omega(t)) \geqslant e^{-\kappa x_\omega(t)},\ t \geqslant T_N.$$

Thus we have $e^{-\kappa x_\omega(t)} \leqslant \gamma_2 e^{-(a-\epsilon)t}$, which implies by $T_N \geqslant 1/R$ that

$$\frac{x_\omega(t)}{t} \geqslant \frac{a - \epsilon - t^{-1}\ln\gamma_2}{\kappa} \geqslant \frac{a - \epsilon - R\ln\gamma_2}{\kappa} = N,\ t \geqslant T_N.$$

Then we obtain (3.35), which completes the proof. □

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